Statistical Methods AEMA 610

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Expectations of Mean Squares, Fixed Effects One-Way ANOVA

When we write out our ANOVA table and ask what Mean Square to divide by what to produce our F-tests, how do we (I) get the Expected Mean Squares that we see? It is not magic; it is simply a question of working through the expectations, by taking our formulae and working out the expectations of each part, using a number of simple, basic principles.

Let us look at a simple One-way, balanced ANOVA. Take our original Completely Randomized Design and for the sake of initial simplicity let us consider that there were no missing values, so that we had 6 treatments and 5 observations for each and every treatment. In general we can say that we have t treatments and r experimental units per treatment.

The 'classical' formulae for such a balanced ANOVA are:

$$\mathrm{SS}_{\mathrm{treatment}} = \sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}$$

$$\mathrm{SS}_{\mathrm{Error}} = \sum_{i=1}^{i=t} \left[\sum_{j=1}^{j=r} Y_{ij}^2 - \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} \right]$$

What are the expectations of these Sums of Squares? Well we have to look at the model that we have just fitted:

$$Y_{ij} = \mu + t_i + e_{ij}$$

We need to look at, and compute, the Expectations of the above Sums of Squares, making use of the above formula for Y_{ij}. Thus:

$$E\left[\mathrm{SS}_{trt}\right] = E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r}Y_{ij}\right)^2}{rt}\right]$$

and

$$E\left[\mathrm{SS}_{residual}\right] = E\left[\sum_{i=1}^{i=t}\left[\sum_{j=1}^{j=r}Y_{ij}^2 - \frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right]\right]$$

We need to compute these expectations one part at a time.

Expectation of Y_{ij}^2

$$E\left[\sum_{j=1}^{j=r}Y_{ij}^2\right] = E\left[\sum_{j=1}^{j=r}(\mu + t_i + e_{ij})^2\right]$$

$$E\left[\sum_{j=1}^{j=r}Y_{ij}^2\right] = E\left[\sum_{j=1}^{j=r}(\mu + t_i + e_{ij})(\mu + t_i + e_{ij})\right]$$

$$E\left[\sum_{j=1}^{j=r}Y_{ij}^2\right] = E\left[\sum_{j=1}^{j=r}(\mu^2 + \mu t_i + \mu e_{ij} + t_i\mu + t_i^2 + t_ie_{ij} + e_{ij}\mu + e_{ij}t_i + e_{ij}e_{ij})\right]$$

$$E\left[\sum_{j=1}^{j=r}Y_{ij}^2\right] = E\left[\sum_{j=1}^{j=r}(\mu^2 + 2\mu t_i + t_i^2 + 2t_ie_{ij} + 2\mu e_{ij} + e_{ij}^2)\right]$$

$$E\left[\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[r\mu^2 + 2r\mu t_i + rt_i^2 + 2\sum_{j=1}^{j=r} t_i e_{ij} + 2\sum_{j=1}^{j=r} \mu e_{ij} + \sum_{j=1}^{j=r} e_{ij}^2\right]$$

$$E\left[\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[r\mu^2 + 2r\mu t_i + rt_i^2 + 2t_i\sum_{j=1}^{j=r} e_{ij} + 2\mu\sum_{j=1}^{j=r} e_{ij} + \sum_{j=1}^{j=r} e_{ij}^2\right]$$

NOTE $E(e_{ij}) = 0$ AND $E(e_{ij}^2) = \sigma_e^2$

$$E\left[\sum_{j=1}^{j=r} Y_{ij}^2\right] = r\mu^2 + 2r\mu t_i + rt_i^2 + 2t_i E\left[\sum_{j=1}^{j=r} e_{ij}\right] + 2\mu E\left[\sum_{j=1}^{j=r} e_{ij}\right] + E\left[\sum_{j=1}^{j=r} e_{ij}^2\right]$$

$$2t_i E\left[\sum_{j=1}^{j=r} e_{ij}\right] = 0$$

$$2\mu E\left[\sum_{j=1}^{j=r} e_{ij}\right] = 0$$

$$E\left[\sum_{j=1}^{j=r} e_{ij}^2\right] = \sum_{j=1}^{j=r} E\left[e_{ij}^2\right] = \sum_{j=1}^{j=r} \sigma_e^2 = r\sigma_e^2$$

$$E\left[\sum_{j=1}^{j=r} Y_{ij}^2\right] = r\mu^2 + 2r\mu t_i + rt_i^2 + r\sigma_e^2$$

Therefore summing over i, *i.e.* $\sum_{i=1}^{i=t}$ we have

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} t_i + r\sum_{i=1}^{i=t} t_i^2 + rt\sigma_e^2$$

Expectation of $(\sum Y_{ij})^2/r$

$$E\left[\frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = E\left[\frac{\left(\sum_{j=1}^{j=r}(\mu + t_i + e_{ij})\right)^2}{r}\right]$$

$$E\left[\frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = E\left[\frac{\left(r\mu + rt_i + \sum_{j=1}^{j=r} e_{ij}\right)^2}{r}\right]$$

$$E\left[\frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = E\left[\frac{\left(r\mu + rt_i + \sum_{j=1}^{j=r} e_{ij}\right)\left(r\mu + rt_i + \sum_{j=1}^{j=r} e_{ij}\right)}{r}\right]$$

$$= E\left[\frac{\left(r^2\mu^2 + r^2\mu t_i + r\mu\sum_{j=1}^{j=r} e_{ij} + r^2\mu t_i + r^2 t_i^2 + rt_i\sum_{j=1}^{j=r} e_{ij} + r\mu\sum_{j=1}^{j=r} e_{ij} +\right)}{r}\right]$$

$$+E\left[\frac{\left(rt_i\sum_{j=1}^{j=r} e_{ij} + \sum_{j=1}^{j=r} e_{ij}\sum_{j=1}^{j=r} e_{ij}\right)}{r}\right]$$

$$E\left[\frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = E\left[r\mu^2 + 2r\mu t_i + rt_i^2 + 2\mu\sum_{j=1}^{j=r} e_{ij} + 2t_i\sum_{j=1}^{j=r} e_{ij} + \frac{\sum_{j=1}^{j=r} e_{ij}\sum_{j=1}^{j=r} e_{ij}}{r}\right]$$

$$E\left[\frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = r\mu^2 + 2r\mu t_i + rt_i^2 + \sigma_e^2$$

$$E\left[\sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = \sum_{i=1}^{i=t} \left(r\mu^2 + 2r\mu t_i + rt_i^2 + \sigma_e^2\right)$$

$$E\left[\sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right] = rt\mu^2 + 2r\mu \sum_{i=1}^{i=t} t_i + r\sum_{i=1}^{i=t} t_i^2 + t\sigma_e^2$$

Expectation of Sums of Squares Error

Thus combining the above two expectations we have

$$E\left[\mathrm{SS}_{residual}\right] = E\left[\sum_{i=1}^{i=t}\left[\sum_{j=1}^{j=r} Y_{ij}^2 - \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right]\right]$$

$$E\left[\mathrm{SS}_{residual}\right] = \left(rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} t_i + r\sum_{i=1}^{i=t} t_i^2 + rt\sigma_e^2\right) - \left(rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} t_i + r\sum_{i=1}^{i=t} t_i^2 + t\sigma_e^2\right)$$

$$E\left[\mathrm{SS}_{residual}\right] = rt\sigma_e^2 - t\sigma_e^2$$

$$E\left[\mathrm{SS}_{residual}\right] = t(r-1)\sigma_e^2$$

Note that the Residual degrees of freedom are $t(r-1)$, thus this shows once again that when we divide the SSE by the residual degrees of freedom (d.f.e.) to compute the Mean Square Error that this will have an expectation of σ_e^2.

Expectation of the Correction Factor for the Mean

$$\text{CF} \;= N\bar{y}^2 = rt\left(\frac{\sum_{i=1}^{i=t}\sum_{j=1}^{j=2} Y_{ij}}{rt}\right)^2 = \frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=2} Y_{ij}\right)^2}{rt}$$

$$E(CF) = E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=2} Y_{ij}\right)^2}{rt}\right]$$

$$E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=2} Y_{ij}\right)^2}{rt}\right] = E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=2}(\mu + t_i + e_{ij})\right)^2}{rt}\right]$$

$$E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=2} Y_{ij}\right)^2}{rt}\right] = E\left[\frac{\left(\sum_{i=1}^{i=t}(r\mu + rt_i + \sum_{j=1}^{j=2} e_{ij})\right)^2}{rt}\right]$$

$$E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=2} Y_{ij}\right)^2}{rt}\right] = E\left[\frac{\left((rt\mu + r\sum_{i=1}^{i=t} t_i + \sum_{i=1}^{i=t}\sum_{j=1}^{j=2} e_{ij})\right)^2}{rt}\right]$$

$$E\left[\frac{\left((rt\mu + r\sum_{i=1}^{i=t} t_i + \sum_{i=1}^{i=t}\sum_{j=1}^{j=2} e_{ij})\right)\left((rt\mu + r\sum_{i=1}^{i=t} t_i + \sum_{i=1}^{i=t}\sum_{j=1}^{j=2} e_{ij})\right)}{rt}\right]$$

$$E\left[\frac{r^2t^2\mu^2 + r^2t\mu\sum_{i=1}^{i=t} t_i + rt\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + r^2t\mu\sum_{i=1}^{i=t} t_i + r^2(\sum_{i=1}^{i=t} t_i)^2 + r\sum_{i=1}^{i=t} t_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}}{rt}\right]$$

$$+\left[\frac{rt\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + r\sum_{i=1}^{i=t} t_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}}{rt}\right]$$

$$= E\left[rt\mu^2 + r\mu\sum_{i=1}^{i=t} t_i + \mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + r\mu\sum_{i=1}^{i=t} t_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2 + \frac{1}{t}\sum_{i=1}^{i=t} t_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right]$$

$$E\left[+\frac{1}{t}\sum_{i=1}^{i=t} t_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \frac{1}{rt}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right]$$

$$= E\left[rt\mu^2 + r\mu\sum_{i=1}^{i=t} t_i + r\mu\sum_{i=1}^{i=t} t_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2 + \frac{1}{rt}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right]$$

$$= rt\mu^2 + 2r\mu E\left[\sum_{i=1}^{i=t} t_i\right] + E\left[\frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2\right] + \frac{rt\sigma_e^2}{rt}$$

$$= rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} t_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2 + \sigma_e^2$$

Expectation of the Sums of Squares for Treatments

$$\text{SS}_{\text{treatment}} = \sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}$$

The expectation of the first term in the above equation we computed as part of the expectation for the Sums of Squares of Errors, and the second part of the above expression is the Correction Factor for the Mean, which we have just examined above. Therefore combining them we obtain:

$$E\left[\ \mathrm{SS}_{\mathrm{treatment}}\right] = E\left[\sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}\right]$$

$$= (rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} t_i + r\sum_{i=1}^{i=t} t_i^2 + t\sigma_e^2) - (rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} t_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2 + \sigma_e^2)$$

$$= r\sum_{i=1}^{i=t} t_i^2 + t\sigma_e^2 - \frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2 - \sigma_e^2$$

$$= r\sum_{i=1}^{i=t} t_i^2 - \frac{r}{t}\left(\sum_{i=1}^{i=t} t_i\right)^2 + (t-1)\sigma_e^2$$

$$= r\sum_{i=1}^{i=t} (t_i - \bar{t})^2 + (t-1)\sigma_e^2$$

Note we divide the Sums of Squares for Treatments by their degrees of freedom (t-1) to compute the Mean Square for Treatments, thus the Expectation of the Mean Square for Treatment will be

$$E\left[\mathrm{MS}_{trt}\right] = E\left[\mathrm{SS}_{trt}\right]/df_{trt}$$

Expectations for ANOVA table

Expectations of Sums of Squares (ANOVA)

Source	d.f.	S.S.	E(SS)
treatments	t-1	SS_{trt}	$r\sum_{i=1}^{i=t}(t_i-\bar{t})^2+(t-1)\sigma_e^2$
Residual	t(r-1)	SS_{error}	$t(r-1)\sigma_e^2$

Expectations of Mean Squares (ANOVA)

Mean Square	E(MS)
$MS_{treatments} = SS_{treatments}/df_{trt}$	$\frac{r}{t-1}\sum_{i=1}^{i=t}(t_i-\bar{t})^2+\sigma_e^2$
$MS_{error} = SS_{error}/df_{error}$	σ_e^2

More simply we can write

Source	d.f.	E(MS)
treatments	t-1	$\sigma_e^2+Q(trt)$
Residual	t(r-1)	σ_e^2

Expectations of Mean Squares, Random Effects One-Way ANOVA

When we write out our ANOVA table and ask what Mean Square to divide by what to produce our F-tests, how do we (I) get the Expected Mean Squares that we see? We can extend the work we did in looking at the expectations of Mean Squares under a fixed effects model. It is not magic; it is simply a question of working through the expectations, by taking our formulae and working out the expectations of each part, using a number of simple, basic principles.

Let us look at a simple One-way, balanced ANOVA, for a Random Effects Model (a so-called Model II). Consider that we have groups of dairy cows, the groups being formed on the basis of sires, so that we are looking at paternal half-sib groups; that is to say for each group, the cows all have the same sire, but their dams are all different. In general we can say that we have t sire groups and r cows (daughters)(experimental units) per sire group.

The 'classical' formulae for such a balanced ANOVA are (see STD):

$$\text{SS}_{\text{sires}} = \sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}$$

$$\text{SS}_{\text{Error}} = \sum_{i=1}^{i=t} \sum_{j=1}^{j=r} Y_{ij}^2 - \sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}$$

What are the expectations of these Sums of Squares? Well we have to look at the model that we have just fitted:

$$Y_{ij} = \mu + s_i + e_{ij}$$

We need to look at, and compute, the Expectations of the above Sums of Squares, making use of the above formula for Y_{ij}. Thus:

$$E\left[\mathrm{SS}_{sires}\right] = E\left[\sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}\right]$$

and

$$E\left[\mathrm{SS}_{residual}\right] = E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2 - \sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right]$$

We need to compute these expectations one part at a time; thus.

$$E\left[\mathrm{SS}_{residual}\right] = E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] - E\left[\sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r}\right]$$

Expectation of $\sum_{ij} Y_{ij}^2$

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}(\mu + s_i + e_{ij})^2\right]$$

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}(\mu + s_i + e_{ij})(\mu + s_i + e_{ij})\right]$$

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}(\mu^2 + \mu s_i + \mu e_{ij} + s_i\mu + s_i^2 + s_i e_{ij} + e_{ij}\mu + e_{ij}s_i + e_{ij}e_{ij})\right]$$

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}(\mu^2 + 2\mu s_i + s_i^2 + 2s_i e_{ij} + 2\mu e_{ij} + e_{ij}^2)\right]$$

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} s_i + r\sum_{i=1}^{i=t} s_i^2 + 2\sum_{i=1}^{i=t} s_i \sum_{j=1}^{j=r} e_{ij} + 2\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}^2\right]$$

then splitting this up into the separate parts, to make things easier

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}^2\right] = E\left[rt\mu^2\right] + E\left[2r\mu\sum_{i=1}^{i=t} s_i\right] + E\left[r\sum_{i=1}^{i=t} s_i^2\right] + E\left[2\sum_{i=1}^{i=t} s_i \sum_{j=1}^{j=r} e_{ij}\right] +$$
$$E\left[2\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right] + E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}^2\right]$$

NOTE $E(e_{ij}) = 0$ AND $E(e_{ij}^2) = \sigma_e^2$

NOTE $E(s_i) = 0$ AND $E(s_i^2) = \sigma_s^2$

$$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}Y_{ij}^2\right] = rt\mu^2 + 0 + rt\sigma_s^2 + 0 + 0 + rt\sigma_e^2$$

therefore $$E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}Y_{ij}^2\right] = rt\mu^2 + rt\sigma_s^2 + rt\sigma_e^2$$

Expectation of $\left[\sum_i(\sum_j Y_{ij})^2/r\right]$

$$E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right] = E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}(\mu + s_i + e_{ij})\right)^2}{r}\right]$$

$$E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right] = E\left[\sum_{i=1}^{i=t}\frac{\left(r\mu + rs_i + \sum_{j=1}^{j=r}e_{ij}\right)^2}{r}\right]$$

$$E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right] = E\left[\sum_{i=1}^{i=t}\frac{\left(r\mu + rs_i + \sum_{j=1}^{j=r}e_{ij}\right)\left(r\mu + rs_i + \sum_{j=1}^{j=r}e_{ij}\right)}{r}\right]$$

$$= E\left[\sum_{i=1}^{i=t}\frac{\left(r^2\mu^2 + r^2\mu s_i + r\mu\sum_{j=1}^{j=r}e_{ij} + r^2\mu s_i + r^2 s_i^2 + rs_i\sum_{j=1}^{j=r}e_{ij} + r\mu\sum_{j=1}^{j=r}e_{ij}+\right)}{r}\right]$$

$$+E\left[\sum_{i=1}^{i=t}\frac{\left(rs_i\sum_{j=1}^{j=r}e_{ij} + \sum_{j=1}^{j=r}e_{ij}\sum_{j=1}^{j=r}e_{ij}\right)}{r}\right]$$

$$E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right]=E\left[rt\mu^2+2r\mu\sum_{i=1}^{i=t}s_i+r\sum_{i=1}^{i=t}s_i^2+2\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}e_{ij}+\right.$$

$$\left.2\sum_{i=1}^{i=t}s_i\sum_{j=1}^{j=r}e_{ij}+\sum_{i=1}^{i=t}\frac{\sum_{j=1}^{j=r}e_{ij}\sum_{j=1}^{j=r}e_{ij}}{r}\right]$$

$$E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right]=rt\mu^2+0+rt\sigma_s^2+t\sigma_e^2$$

therefore $$E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right]=rt\mu^2+rt\sigma_s^2+t\sigma_e^2$$

Expectation of Sums of Squares Error

Thus combining the above two expectations we have

$$E\left[\mathrm{SS}_{residual}\right]=E\left[\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}Y_{ij}^2\right]-E\left[\sum_{i=1}^{i=t}\frac{\left(\sum_{j=1}^{j=r}Y_{ij}\right)^2}{r}\right]$$

$$E\left[\mathrm{SS}_{residual}\right]=\left[rt\mu^2+rt\sigma_s^2+rt\sigma_e^2\right]-\left[rt\mu^2+rt\sigma_s^2+t\sigma_e^2\right]$$

$$E\left[\mathrm{SS}_{residual}\right]=rt\sigma_e^2-t\sigma_e^2$$

therefore $\quad E\left[\mathrm{SS}_{residual}\right] = t(r-1)\sigma_e^2$

Note that the Residual degrees of freedom are $t(r-1)$, thus this shows once again that when we divide the SSE by the residual degrees of freedom (d.f.e.) to compute the Mean Square Error that this will have an expectation of σ_e^2.

Expectation of the Correction Factor for the Mean

$$\mathrm{CF} \; = N\bar{y}^2 = rt\left(\frac{\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}}{rt}\right)^2 = \frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}\right)^2}{rt}$$

$$E(CF) = E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}\right)^2}{rt}\right]$$

$$E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}\right)^2}{rt}\right] = E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=r}(\mu + s_i + e_{ij})\right)^2}{rt}\right]$$

$$E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}\right)^2}{rt}\right] = E\left[\frac{\left(\sum_{i=1}^{i=t}(r\mu + rs_i + \sum_{j=1}^{j=r} e_{ij})\right)^2}{rt}\right]$$

$$E\left[\frac{\left(\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} Y_{ij}\right)^2}{rt}\right] = E\left[\frac{\left((rt\mu + r\sum_{i=1}^{i=t} s_i + \sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij})\right)^2}{rt}\right]$$

$$E\left[\frac{\Big((rt\mu + r\sum_{i=1}^{i=t} s_i + \sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij})\Big)\Big((rt\mu + r\sum_{i=1}^{i=t} s_i + \sum_{i=1}^{i=t}\sum_{j=1}^{j=2} e_{ij})\Big)}{rt}\right]$$

$$E\left[\frac{r^2t^2\mu^2 + r^2t\mu\sum_{i=1}^{i=t} s_i + rt\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + r^2t\mu\sum_{i=1}^{i=t} s_i + r^2(\sum_{i=1}^{i=t} s_i)^2 + r\sum_{i=1}^{i=t} s_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}}{rt}\right]$$

$$+\left[\frac{rt\mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + r\sum_{i=1}^{i=t} s_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}}{rt}\right]$$

$$= E\left[rt\mu^2 + r\mu\sum_{i=1}^{i=t} s_i + \mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + r\mu\sum_{i=1}^{i=t} s_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} s_i\right)^2 + \frac{1}{t}\sum_{i=1}^{i=t} s_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \mu\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right]$$

$$E\left[+\frac{1}{t}\sum_{i=1}^{i=t} s_i\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij} + \frac{1}{rt}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right]$$

$$= E\left[rt\mu^2 + r\mu\sum_{i=1}^{i=t} s_i + r\mu\sum_{i=1}^{i=t} s_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} s_i\right)^2 + \frac{1}{rt}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\sum_{i=1}^{i=t}\sum_{j=1}^{j=r} e_{ij}\right]$$

$$= rt\mu^2 + 2r\mu E\left[\sum_{i=1}^{i=t} s_i\right] + E\left[\frac{r}{t}\left(\sum_{i=1}^{i=t} s_i\right)^2\right] + \frac{rt\sigma_e^2}{rt}$$

$$= rt\mu^2 + 2r\mu\sum_{i=1}^{i=t} s_i + \frac{r}{t}\left(\sum_{i=1}^{i=t} s_i\right)^2 + \sigma_e^2$$

$$= rt\mu^2 + 0 + \frac{r}{t}t\sigma_s^2 + \sigma_e^2$$

$$= rt\mu^2 + r\sigma_s^2 + \sigma_e^2$$

Expectation of the Sums of Squares for Treatments

$$\text{SS}_{\text{treatment}} = \sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}$$

The expectation of the first term in the above equation we computed as part of the expectation for the Sums of Squares of Errors, and the second part of the above expression is the Correction Factor for the Mean, which we have just examined above. Therefore combining them we obtain:

$$E\left[\ \text{SS}_{\text{treatment}}\right] = E\left[\sum_{i=1}^{i=t} \frac{\left(\sum_{j=1}^{j=r} Y_{ij}\right)^2}{r} - \frac{\left(\sum_{i=1,j=1}^{i=t,j=r} Y_{ij}\right)^2}{rt}\right]$$

$$= \left(rt\mu^2 + rt\sigma_s^2 + t\sigma_e^2\right) - \left(rt\mu^2 + r\sigma_s^2 + \sigma_e^2\right)$$

$$= rt\sigma_s^2 + t\sigma_e^2 - r\sigma_s^2 - \sigma_e^2$$

$$= rt\sigma_s^2 - r\sigma_s^2 + t\sigma_e^2 - \sigma_e^2$$

$$= (t-1)r\sigma_s^2 + (t-1)\sigma_e^2$$

Note we divide the Sums of Squares for Treatments by their degrees of freedom (t-1) to compute the Mean Square for Treatments, thus the Expectation of the Mean Square for Treatment will be

$$E\left[\mathrm{MS}_{trt}\right] = E\left[\mathrm{SS}_{trt}\right]/df_{trt}$$

Expectations for ANOVA table

Expectations of Sums of Squares (ANOVA)

Source	d.f.	S.S.	E(SS)
treatments	t-1	SS_{trt}	$(t-1)r\sigma_s^2 + (t-1)\sigma_e^2$
Residual	t(r-1)	SS_{error}	$t(r-1)\sigma_e^2$

Expectations of Mean Squares (ANOVA)

Mean Square	E(MS)
$\mathrm{MS}_{treatments} = \mathrm{SS}_{treatments}/\mathrm{df}_{trt}$	$\sigma_e^2 + r\sigma_s^2$
$\mathrm{MS}_{error} = \mathrm{SS}_{error}/\mathrm{df}_{error}$	σ_e^2